Math for Management, Winter 2023

List 3

Matrices, systems of linear equations

A **matrix** is a grid of numbers. The **dimensions** of a matrix are written in the format "$m \times n$", spoken as "m by n", where m is the number of rows and n is the number of columns (write both numbers; do not multiply them).

43. Give the dimensions of the following matrices:

(a) $\begin{bmatrix} -92 & 8 \\ -78 & -67 \end{bmatrix}$ $\boxed{2 \times 2}$

(b) $\begin{bmatrix} -36 \\ 72 \\ -12 \end{bmatrix}$ $\boxed{3 \times 1}$

(c) $\begin{bmatrix} 75 & 89 & 50 \\ -5 & -81 & 34 \end{bmatrix}$ $\boxed{2 \times 3}$

(d) $\begin{bmatrix} -13 & -63 & -5 \\ 0 & -66 & \frac{1}{2} \\ 31 & \frac{5}{22} & \frac{8}{11} \end{bmatrix}$ $\boxed{3 \times 3}$

(e) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\boxed{2 \times 2}$

(f) $\begin{bmatrix} 58 & -65 & 40 & 8 & -1 & 26 \\ -74 & 58 & -92 & -4 & -21 & 74 \end{bmatrix}$ $\boxed{2 \times 6}$

In order to for the matrix product MN to exist (that is, for it to be possible to do the multiplication MN) it must be that the number of columns of M is equal to the number of rows of N.

44. If A is a 2×2 matrix, B is a 3×3 matrix, and C is a 3×2 matrix, which of the following exist?

(a) AA exists (2×2)

(b) AB doesn't exist

(c) AC doesn't exist

(d) BA doesn't exist

(e) BB exists (3×3)

(f) BC exists (3×2)

(g) CA exists (3×2)

(h) CB doesn't exist

(i) CC doesn't exist

(j) ABC doesn't exist

(k) BCA exists (3×2)

(ℓ) ACA doesn't exist

(m) $A^\top C$ doesn't exist

(n) $AC^\top$ exists (2×3)

(o) $C^\top C$ exists (2×2)

(p) $AB^\top CAC^\top$ doesn't exist

45. (a) Calculate $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}$. $\boxed{\begin{bmatrix} 7 & 24 \\ 3 & -72 \end{bmatrix}}$ Previous file had $\begin{bmatrix} 7 & 24 \\ -1 & 96 \end{bmatrix}$.

(b) Calculate $\begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}$. $\boxed{\begin{bmatrix} 3 & 6 \\ 62 & -92 \end{bmatrix}}$ Previous file had $\begin{bmatrix} 3 & 6 \\ 62 & -68 \end{bmatrix}$.

(b) Compare your answers to parts (a) and (b). They are not equal. In general, MN and NM can be different matrices.

The **transpose** of a matrix M, written $M^\top$, swaps the rows and columns. For example, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

46. Compute the following:

(a) $\begin{bmatrix} 1 & 14 & 8 \\ 6 & -1 & 14 \\ 5 & 11 & -3 \end{bmatrix} + \begin{bmatrix} 11 & 2 & 10 \\ -2 & 14 & 8 \\ -2 & 3 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} 12 & 16 & 18 \\ 4 & 13 & 22 \\ 3 & 14 & -7 \end{bmatrix}}$

(b) $\begin{bmatrix} 1 & 14 & 8 \\ 6 & -1 & 14 \\ 5 & 11 & -3 \end{bmatrix} - \begin{bmatrix} 11 & 2 & 10 \\ -2 & 14 & 8 \\ -2 & 3 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} -10 & 12 & -2 \\ 8 & -15 & 6 \\ 7 & 8 & 1 \end{bmatrix}}$

(c) $3\begin{bmatrix} 0 & -4 & 0 \\ -1 & -1 & 3 \\ -2 & 5 & 14 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -12 & 0 \\ -3 & -3 & 9 \\ -6 & 15 & 42 \end{bmatrix}}$

(d) $\frac{1}{6}\begin{bmatrix} 9 & 14 \\ 6 & 10 \end{bmatrix} = \boxed{\begin{bmatrix} 3/2 & 7/3 \\ 1 & 5/3 \end{bmatrix}}$

(e) $\begin{bmatrix} 8 & 5 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} 33 \\ -25 \end{bmatrix}}$

(f) $\begin{bmatrix} 9 & 8 \\ -2 & 5 \end{bmatrix}^{\top}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 6 \\ 46 \end{bmatrix}}$

(g) $\begin{bmatrix} -5 & 5 & 7 \\ -2 & -3 & 2 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \\ 8 \end{bmatrix} = \boxed{\begin{bmatrix} 86 \\ -17 \\ 35 \end{bmatrix}}$

(h) $\begin{bmatrix} 4 & 8 & 0 \\ -3 & 3 & -3 \\ 8 & 5 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix} = \boxed{\begin{bmatrix} 24 \\ -3 \\ -10 \end{bmatrix}}$

(i) $\begin{bmatrix} 4 & 8 & 0 \\ -3 & 3 & -3 \\ 8 & 5 & -2 \end{bmatrix}^{\top}\begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 8 \\ 8 & 3 & 5 \\ 0 & -3 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix} = \boxed{\begin{bmatrix} 36 \\ 31 \\ -26 \end{bmatrix}}$

47. Compute $\begin{bmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}. = \boxed{\begin{bmatrix} 2-\sqrt{2} \\ 6\sqrt{2} \\ 2+\sqrt{2} \end{bmatrix}}$

48. Compute the following, if they exist:

(a) $\begin{bmatrix} 9 & -4 \\ -5 & -5 \end{bmatrix}\begin{bmatrix} 8 & 1 \\ 0 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} 72 & 21 \\ -40 & 10 \end{bmatrix}}$

(b) $\begin{bmatrix} 4 & 5 & 22 \\ 8 & -13 & 4 \end{bmatrix}\begin{bmatrix} 19 & 0 & 35 & 6 \\ 0 & 2 & 2 & 6 \\ 9 & 1 & 19 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 274 & 32 & 568 & 32 \\ 188 & -22 & 330 & -34 \end{bmatrix}}$

(c) $\begin{bmatrix} 19 & 0 & 35 & 6 \\ 0 & 2 & 2 & 6 \\ 9 & 1 & 19 & -1 \end{bmatrix}\begin{bmatrix} 4 & 5 & 22 \\ 8 & -13 & 4 \end{bmatrix}$ $\boxed{\text{does not exist}}$

(d) $\begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 & 2 & 7 \\ 3 & -4 & -1 & 1 & 8 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 6 & -3 & 6 & 21 \\ 6 & -4 & -4 & 6 & 30 \end{bmatrix}}$

(e) $\begin{bmatrix} -2 & -4 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 2 & 8 \end{bmatrix} = \boxed{\begin{bmatrix} 72 & 48 \\ -36 & -24 \end{bmatrix}}$

(f) $\begin{bmatrix} -4 & -3 & -5 \\ 24 & 6 & 29 \end{bmatrix} \begin{bmatrix} 4 & 13 & 0 \\ 2 & -26 & 9 \end{bmatrix}$ does not exist

(g) $\begin{bmatrix} -4 & -3 & -5 \\ 24 & 6 & 29 \end{bmatrix} \begin{bmatrix} 4 & 13 & 0 \\ 2 & -26 & 9 \end{bmatrix}^\top = \begin{bmatrix} -4 & -3 & -5 \\ 24 & 6 & 29 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 13 & -26 \\ 0 & 9 \end{bmatrix} = \boxed{\begin{bmatrix} -55 & 25 \\ 174 & 153 \end{bmatrix}}$

49. (a) Calculate $\left(\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix} \right)^\top = \begin{bmatrix} 7 & 24 \\ -1 & -96 \end{bmatrix}^\top = \boxed{\begin{bmatrix} 7 & -1 \\ 24 & -96 \end{bmatrix}}$

(b) Calculate $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}^\top \begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}^\top = \begin{bmatrix} 1 & 5 \\ 2 & -8 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 9 & 12 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 62 \\ 6 & -92 \end{bmatrix}}$

(c) Calculate $\begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}^\top \begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}^\top = \begin{bmatrix} 3 & 2 \\ 9 & 12 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 2 & -8 \end{bmatrix} = \boxed{\begin{bmatrix} 7 & -1 \\ 24 & -96 \end{bmatrix}}$

(d) Compare your answers to parts (a) and (b). They are not equal.

(e) Compare your answers to parts (a) and (c). They are equal! In general, $(AB)^\top = B^\top A^\top$.

50. Compute the following:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 3 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} 8 & 2 \\ 3 & -3 \end{bmatrix}}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 14 & 21 \\ -11 & 23 \end{bmatrix} = \boxed{\begin{bmatrix} 14 & 21 \\ -11 & 23 \end{bmatrix}}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 99 & \frac{1}{10} \\ -37 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 99 & \frac{1}{10} \\ -37 & 2 \end{bmatrix}}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & -2 & 1 \\ 5 & 2 & 5 \\ 7 & 4 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -4 & -2 & 1 \\ 5 & 2 & 5 \\ 7 & 4 & 1 \end{bmatrix}}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 15 & 11 & -15 \\ 17 & 10 & -8 \\ 0 & 0 & -13 \end{bmatrix} = \boxed{\begin{bmatrix} 15 & 11 & -15 \\ 17 & 10 & -8 \\ 0 & 0 & -13 \end{bmatrix}}$

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix} = \boxed{\begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix}}$

(g) $\begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix}}$

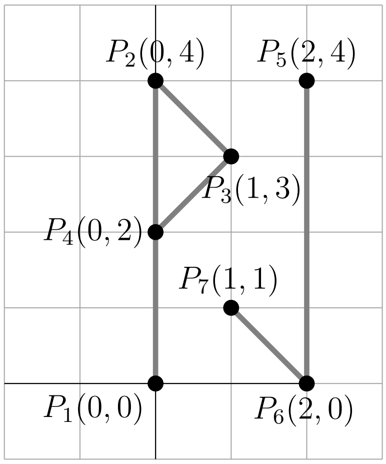
(h) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -39 & -66 & 84 & 66 & -10 \\ -47 & -5 & 17 & -59 & -3 \\ -94 & -90 & -5 & 86 & 31 \\ 25 & 80 & 0 & 35 & 19 \\ -72 & 40 & 99 & 48 & 57 \end{bmatrix} = \boxed{\begin{bmatrix} -39 & -66 & 84 & 66 & -10 \\ -47 & -5 & 17 & -59 & -3 \\ -94 & -90 & -5 & 86 & 31 \\ 25 & 80 & 0 & 35 & 19 \\ -72 & 40 & 99 & 48 & 57 \end{bmatrix}}$

51. For each of the points P_1 through P_7, calculate

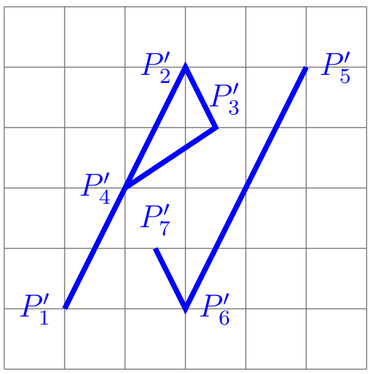
$$P_i{}' = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} P_i.$$

(For example, for $P_5{}' = \left[\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 4 \\ 4 \end{smallmatrix}\right]$.) Plot the points $P_1{}', \ldots, P_7{}'$ on a new grid. Connect $P_1{}' \to P_2{}' \to P_3{}' \to P_4{}'$ with line segments, and connect $P_5{}' \to P_6{}' \to P_7{}'$.

Congratulations. You can write in italics!



$T(P_1) = (0,0)$ $T(P_2) = (2,4)$ $T(P_3) = (5/2,3)$ $T(P_4) = (1,2)$
$T(P_5) = (4,4)$ $T(P_6) = (2,0)$ $T(P_7) = (3/2,1)$



52. If $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix} M = \begin{bmatrix} 8 & 25 & 12 \\ 14 & 45 & 22 \end{bmatrix}$, what are the dimensions of matrix M? $\boxed{2 \times 3}$

53. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 5 & 2 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$. Write all the products of two matrices from this list that exist (e.g., AA exists, but AC does not).

There are 9 valid products of this form: $\boxed{AA, AB, BA, BB, CD, DC, DE, EA, EB}$

54. For each of the following equations, either give the dimensions of the matrix M or state that such a matrix does not exist. (You do *not* have to solve for M.)

(a) $M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $\boxed{2 \times 1}$

(b) $M = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ $\boxed{\text{doesn't exist}}$

(c) $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\boxed{\text{doesn't exist}}$

(d) $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\boxed{3 \times 2}$

(e) $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\boxed{2 \times 1}$

(f) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} M \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\boxed{1 \times 3}$

(g) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} M \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\boxed{\text{doesn't exist}}$

(h) $\begin{bmatrix} 2 & -8 \\ 1 & 5 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 & 11 & 4 \\ -2 & -8 & 6 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \\ -9 \end{bmatrix} \begin{bmatrix} \frac{2}{7} & 1 & \frac{4}{7} \end{bmatrix} = M$

Dimensions $(3 \times 2)(2 \times 5)(5 \times 1)(1 \times 3)$ leads to $\boxed{3 \times 3}$.

Earlier versions of Tasks 55 and 56 involved "determinants", which are not part of MAT 1448.

55. Suppose M is a 5×12 matrix. Can there be a matrix N such that both MN and NM exist? If so, can anything be said about the dimensions of N? $\boxed{N \text{ must be } 12 \times 5}$

56. Calculate $\begin{bmatrix} 11 & \frac{9}{2} \\ -2 & 21 \end{bmatrix}^2$ and $\begin{bmatrix} -16 & 18 \\ -8 & 24 \end{bmatrix}^2$ and compare the answers.

Both are $\boxed{\begin{bmatrix} 122 & 144 \\ -64 & 432 \end{bmatrix}}$. In general there can be many, many solution to $X^2 = M$.

The $\boldsymbol{n \times n}$ **identity matrix** is the matrix I (also written I_n or $I_{n\times n}$) such that

$$IM = MI = M$$

for any $n \times n$ matrix M. It has 1 along the main diagonal and 0 everywhere else.

57. (a) Multiply $\begin{bmatrix} 0 & -4 & 2 \\ 10 & -1 & 0 \\ 1 & 12 & -6 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 30 & -1 & 10 \\ \frac{121}{2} & -2 & 20 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

(b) Multiply $\begin{bmatrix} 0 & -4 & 2 \\ 10 & -1 & 0 \\ 1 & 12 & -6 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 30 & -1 & 10 \\ \frac{121}{2} & -2 & 20 \end{bmatrix} \begin{bmatrix} 6 & 4 & 9 \\ 0 & -3 & -2 \\ 4 & 6 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 4 & 9 \\ 0 & -3 & -2 \\ 4 & 6 & -4 \end{bmatrix}}$

The **inverse matrix** of a matrix M is written M^{-1} (spoken as "M inverse") and it is the unique matrix for which $M^{-1}\,M = I$ and $M\,M^{-1} = I$. For a 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

For any square matrix, the inverse can be found by carefully applying "row operations" to the "augmented matrix" $[\,M\,|\,I\,]$ until it becomes $[\,I\,|\,M^{-1}\,]$.

58. Find $\begin{bmatrix} 5 & 4 \\ 1 & -2 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{1}{14} & -\frac{5}{14} \end{bmatrix}}$ and $\begin{bmatrix} 0 & -4 & 2 \\ 10 & -1 & 0 \\ 1 & 12 & -6 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} 3 & 0 & 1 \\ 30 & -1 & 10 \\ \frac{121}{2} & -2 & 20 \end{bmatrix}}$ from Task 57a.

59. Find the inverses of the following matrices. Compare the answers of parts (e) and (f).

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}}$

(b) $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} \frac{9}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}}$

(c) $\begin{bmatrix} 8 & 4 \\ 6 & 3 \end{bmatrix}^{-1}$ does not exist

(d) $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} \frac{5}{17} & \frac{1}{17} \\ -\frac{2}{17} & \frac{3}{17} \end{bmatrix}}$

(e) $\begin{bmatrix} 3 & b \\ 2 & 5 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} \frac{5}{15-2b} & -\frac{b}{15-2b} \\ -\frac{2}{15-2b} & \frac{3}{15-2b} \end{bmatrix} \text{ if } b \neq \frac{15}{2}}$

(f) $\begin{bmatrix} 3 & 2 \\ b & 5 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} \frac{5}{15-2b} & -\frac{2}{15-2b} \\ -\frac{b}{15-2b} & \frac{3}{15-2b} \end{bmatrix} \text{ if } b \neq \frac{15}{2}}$

(g) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 9 \\ 5 & 6 & 8 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} -\frac{22}{25} & \frac{6}{25} & \frac{7}{25} \\ \frac{29}{25} & -\frac{17}{25} & \frac{1}{25} \\ -\frac{8}{25} & \frac{9}{25} & -\frac{2}{25} \end{bmatrix}}$

☆(h) $\begin{bmatrix} 3 & 1 & 3 & 3 \\ 1 & 2 & 4 & 0 \\ 2 & 0 & 2 & 2 \\ 4 & 2 & 1 & 3 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} -2 & \frac{1}{3} & 2 & \frac{2}{3} \\ 1 & 0 & -\frac{3}{2} & 0 \\ 0 & \frac{1}{6} & \frac{1}{4} & -\frac{1}{6} \\ 2 & -\frac{1}{2} & -\frac{7}{4} & -\frac{1}{2} \end{bmatrix}}$

60. Find the matrix M from Task 52. $\boxed{\begin{bmatrix} 1 & 0 & -1 \\ 1 & 5 & 3 \end{bmatrix}}$

61. Solve the following matrix equations:

(a) $X\begin{bmatrix} -1 & 1 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}$. So $XI = \begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & -4 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} 11 & 3 \\ -24 & -7 \end{bmatrix}}$

(b) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$. $\boxed{X = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}}$

(c) $\left(\begin{bmatrix} 0 & 3 \\ 5 & -2 \end{bmatrix} + 4X\right)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. $X = \frac{1}{4}(B^{-1} - A) = \boxed{\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{8} & \frac{3}{8} \end{bmatrix}}$

(d) $3\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = X^{\top}\begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$. $\boxed{X = \begin{bmatrix} -30 & 33 \\ 51/2 & -57/2 \end{bmatrix}}$

Earlier versions of Tasks 62 and parts of Task 63 also involved determinants.

☆62. Find a matrix X for which $\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} X = \begin{bmatrix} 10 & 4 \\ 10 & 4 \end{bmatrix}$. Any matrix $X = \begin{bmatrix} 5 & 2 \\ c & d \end{bmatrix}$ with any bottom row will work.

63. For each of the following, does an inverse matrix exist?

(a) the 3×3 identity matrix.

(b) a 3×5 matrix where every number in the matrix is 1.

(c) a 4×4 matrix where every number in the matrix is 1.

(d) a 4×4 matrix where every number in the matrix is 0.

(e) a 2×2 matrix with $a_{ij} = i + j$.

$\boxed{\text{Only } A \text{ and } E \text{ have an inverse}}$

64. Use inverse matrices to solve these systems:

(a) $2x - y = 3,\ 3x + y = 2$

$$\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ leads to } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so $\boxed{x = 1, y = -1}$

(b) $x + 2y = 0,\ 2x - y = 5$ $\boxed{x = 2, y = -1}$

(c) $\begin{cases} x + y + z = 5 \\ 2x + 2y + z = 3 \\ 3x + 2y + z = 1 \end{cases}$ $\boxed{x = -2, y = -2, z = 7}$

(d) $\begin{cases} x + y + z = 4 \\ 2x - 3y + 5z = -5 \\ -x + 2y - z = 2 \end{cases}$ $\boxed{x = 3, y = 3, z = -1}$

65. Use the Gauss method to solve the systems of linear equations from Task 64.

66. Solve the following systems of equations:

(a) $\begin{cases} x + 2y + 3z = 14 \\ 4x + 3y - z = 7 \\ x - y + z = 2 \end{cases}$ $\boxed{x = 1, y = 2, z = 3}$

(b) $\begin{cases} 3x + 4y + z + 2t = 3 \\ 6x + 8y + 2z + 5t = 7 \\ 9x + 12y + 3z + 10t = 13 \end{cases}$

$\boxed{t = 1,\ z = 1 - 3x - 4y, \text{ where } x \text{ and } y \text{ can be anything}}$

(c) $\begin{cases} 3x - 5y + 2z + 4t = 2 \\ 7x - 4y + z + 3t = 5 \\ 5x + 7y - 4z - 6t = 3 \end{cases}$ $\boxed{\text{no solution}}$

(e) $\begin{cases} 3x + 2y + 2z + 2t = 2 \\ 2x + 3y + 2z + 5t = 3 \\ 9x + y + 4z - 5t = 1 \\ 2x + 2y + 3z + 4t = 5 \\ 7x + y + 6z - t = 7 \end{cases}$ $\boxed{x = \frac{8t}{7} - \frac{6}{7}, y = \frac{1}{7} - \frac{13t}{7}, z = \frac{15}{7} - \frac{6t}{7}}$